1. CALCULUS OF VARIATIONS

1.0 Introduction

The usual condition that a function of n variables, $f(x_1, x_2, --- x_n)$ has a stationary value is stated as:

$$df = o$$

or

$$\text{grad} \quad f = o$$

or

$$\frac{\partial f}{\partial x_i} = o \tag{1.1}$$

The calculus of variations is concerned with investigating "extreme" --or, more generally, "stationary" --properties of a function. However, we are in this case presented with a function of a function and it is necessary to develop new methods for investigating the stationary properties. Our procedure will be analogous to the procedure by which equations (1.1) are obtained.

1.1 Brachistachrone Problem

Attention to problems of the variational type was called by Bernoulli in 1696 who stated the "BRACHISTOCHRONE" problem (shortest time problem) as follows:

In a vertical xy plane, a point A is to be joined to a point B, which is lower (but not directly below) than A, by a smooth curve $y = u(x)$ in such a way that a frictionless particle will slide from A to B along y in the shortest possible time.

Since there is no friction, the equation of motion is:

$$\frac{1}{2} mv^2 + mgh = \text{constant} \tag{1.2}$$

If we take as an initial condition that $v = o$ at $h = o$, then:

$$v = \sqrt{-2gh} \tag{1.3}$$

Let us take $y = - h$

so: $v = \sqrt{2gy}$ (1.4)

since $v = \frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt}$ (1.5)

and $T = \int_0^T dt = \int_{x_A}^{x_B} \frac{1}{v}\frac{ds}{dx} \quad dx$

also $ds^2 = dx^2 + dy^2$ (1.6)

and $\frac{ds}{dx} = \sqrt{1 + \frac{(dy)^2}{(dx)}}$ (1.7)

Finally, the general expression relating the sliding time "T" and the "slide path" Y is:

$$T = \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \sqrt{\frac{1 + (y')^2}{y}} \quad dx \tag{1.8}$$

where we have written y' for (dy/dx)

We now state the brachistochrone problem as:

Among all curves $y = \emptyset (x)$ which are continuously differentiable and which pass through the points $(x_A, 0)$ and (x_B, y_B), find that particular $y (x)$ for which the integral

$$T = \frac{1}{\sqrt{2g}} \int_{x_A}^{x_B} \sqrt{\frac{1 + y'^2}{y}} \quad dx \tag{1.8}$$

will have the least possible value.

This is the problem that started it all. It is an example of the simplest kind of problem of the calculus of variations. Note the essential difference between this problem and the ordinary maximum--minimum problems. Here the value of the integral, "I", depends on the behavior of y throughout the entire interval of x. The value of "I" cannot be specified by a finite number of finite variables, but rather must be specified by prescribing a continuous function.

Problems of this type are of great importance in trajectory analysis. For example, consider a very frequently occurring problem:

How should the attitude of a given booster be programmed in order to boost a satellite to orbital altitude with maximum horizontal component of velocity?

Apparently, the horizontal velocity attained will depend on the entire attitude history, or program, during powered flight.

The above problems will be treated in some detail after we have developed some of the methods of the calculus of variations.

1.2 Development of Euler's Equation

Consider the following general problem:

Among all twice differentiable curves, $y = \phi\ (x)$ find that particular $\phi\ (x)$ which passes through the points (x_o, y_o) and (x_1, x_1) and which causes the integral*

$$I\{\phi\} = \int_{x_o}^{x_1} F\ (x, y, y')\ dx \qquad (1.9)$$

to assume a stationary value.

Our procedure for finding the particular $\phi\ (x)$ which satisfies the requirements is analogous to the procedure for finding the extreme value of a function.

We assume that $\phi = U\ (x)$ is the particular curve desired. Then, any other curve which is "admissible", for example: (see following page)

* The requirement for continuous derivatives will be examined later.

$$\phi = u\ (x) + \epsilon\eta(x)^{*} \qquad (1.10)$$

must cause the integral "I" to move from the extreme value. For example, if u is the admissible function which minimizes "I" then $u + \epsilon\eta\ (x)$ will cause "I" to increase. Here ϵ is a constant and $\eta(x)$ is a function with continuous first and second derivations which vanishes at x_o and x_1, that is:

$$\eta\ (x_o) = \eta\ (x_1) = o \qquad (1.11)$$

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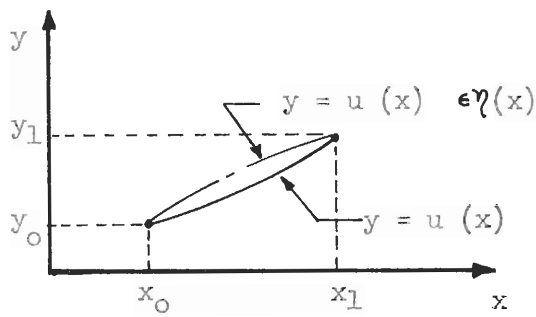

Figure 1

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We have:

$$y = \phi\ (x) = u \quad \epsilon\eta(x)$$

$$(1.12)$$

and $$y' = \phi' = u' \quad \epsilon\eta'(x)$$

and

$$"I" = \int_{x_o}^{x_1} F\ (x,\ y,\ y')\ dx = \int_{x_o}^{x_1} F\ (x,\ \ \epsilon\eta.\ u'\ \ \epsilon\eta')\ dx$$

This integral is a function of ϵ :

$$\Psi\ (\epsilon) = I = \int_{x_o}^{x_1} F\ (x,\ u\ \ \epsilon\eta.\ u' + \epsilon\eta'\ )\ dx \qquad (1.13)$$

Now we can state more clearly the condition for which u (x) is the function which causes "I" to be stationary. Clearly if this be true,

*The expression $\phi(x)$ is called the variation of the function u (x).

then:

$$\frac{d\psi}{d\epsilon} = 0 \text{ at } \epsilon = 0 \tag{1.14}$$

Remembering that $y = u + \epsilon\eta$, we compute $\frac{d\psi}{d\epsilon}$ as:

$$\frac{d\psi}{d\epsilon} = \frac{d}{d\epsilon} \int_{x_0}^{x_1} F(x, y, y')\, dx$$

$$\frac{d\psi}{d\epsilon} = \int_{x_0}^{x_1} \frac{dF}{d\epsilon}(x, y, y')\, dx$$

$$\frac{d\psi}{d\epsilon} = \int_{x_0}^{x_1} \left(\frac{\partial F}{\partial y} \frac{dy}{d\epsilon} + \frac{\partial F}{\partial y'} \frac{dy'}{d\epsilon} \right) dx$$

$$\frac{d\psi}{d\epsilon} = \int_{x_0}^{x_1} \left(\eta \frac{\partial F}{\partial y} + \eta' \frac{\partial F}{\partial y'} \right) dx \tag{1.15}$$

The conditions on the functions u and η assure that it is permissable to interchange the order of integration and differentiation.

Applying the stationary conditions to equation (1.15) we obtain:

$$\left.\frac{d\psi}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1} \left(\eta \frac{\partial F}{\partial u} + \eta' \frac{\partial F}{\partial u'} \right) dx = 0 \tag{1.16}$$

Equation (1.16) expresses the condition for which $y = u(x)$ causes "I" to be stationary.

Equation (1.16) can be put in a more convenient form if we integrate the last half of the expression by parts.

$$\int_{x_0}^{x_1} \eta' \frac{\partial F}{\partial u'}\, dx = \eta \frac{\partial F}{\partial u'} \Big]_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx \tag{1.17}$$

Since we required by (1.11) that $\eta(x_0) = \eta(x_1) = 0$, the first term is zero, thus:

$$\int_{x_0}^{x_1} \eta' \frac{\partial F}{\partial u'}\, dx = -\int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx \tag{1.18}$$

substituting this into (1.16):

$$\left.\frac{d\psi}{d\epsilon}\right|_{\epsilon=0} = \int_{x_0}^{x_1} \eta \left[\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right] dx = 0 \qquad (1.19)$$

Now, the function $\eta(x)$ is arbitrary and the quantity $\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)$ is independent of $\eta(x)$. The integral in equation (1.19) cannot possibly be zero for all arbitrary values of $\eta(x)$ unless the total integrand of equation (1.19) is exactly zero.

This requires that:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \qquad (1.20)$$

Equation (1.20) is known as EULER'S EQUATION as it was discovered by Euler in 1744. Expanded and written out in full, equation (1.20) becomes:

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = \frac{\partial F}{\partial u} - \frac{\partial^2 F}{\partial u \partial u'} u' - \frac{\partial^2 F}{\partial u'^2} u'' - \frac{\partial^2 F}{\partial x \partial u'} \qquad (1.20)$$

Euler's equation is an ordinary differential equation of the second order. The solutions are called extremals of the variational problem.

The particular extremal, (if any), which satisfies the boundary conditions is the desired solution of the variational problem. Note that, since (1.20) is a second order equation, two arbitrary constants of integration are available for satisfying boundary conditions.

1.3 Solution of the Brachistochrone

For the simple Brachistochrone problem we seek to minimize the integral:

$$I = \frac{1}{\sqrt{2g}} \int_{x_0}^{x_1} \sqrt{\frac{1 + y'^2}{y}}\, dx \qquad (1.8)$$

Applying Euler's equation we obtain:

$$-\tfrac{1}{2} y^{-3/2}\sqrt{1 + y'^2} - \frac{d}{dx}\left[\frac{y'}{\sqrt{y}}\left(1 + y'^2\right)^{-\frac{1}{2}}\right] = 0$$

The solutions of this equation are the brachistochrones. In this case, however, the Euler equation can be simplified directly. Note that the explicit dependence of "F" on x is missing. When this is true:

$$\frac{d}{dx}\left[y' \frac{\partial F}{\partial y'} - F\right] = 0 \qquad (1.21)$$

and

$$y' \frac{\partial F}{\partial y'} - F = \text{constant} \qquad (1.22)$$

This relation, equation (1.22), is then an integral of Euler's differential equation for the special case when the independent variable x does not appear explicitly in the variational problem.

The proof of this follows by forming the derivative

$$\frac{d}{dx}(\quad) \text{ of equation (1.22)}$$

$$\frac{d}{dx}\left[y' \frac{\partial F}{\partial y'} - F\right] = y'' \frac{\partial F}{\partial y'} + y' \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - y' \frac{\partial F}{\partial y} - y'' \frac{\partial F}{\partial y'}$$

or

$$\frac{d}{dx}\left[y' \frac{\partial F}{\partial y'} - F\right] = y'\left[\frac{d}{dx} \frac{\partial F}{\partial y'} - \frac{\partial F}{\partial y}\right]$$

However

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Since this is the basic Euler equation.

Thus:

$$\frac{d}{dx}\left[y' \frac{\partial F}{\partial y'} - F\right] = 0$$

Hence, every solution of Euler equation for this case gives the result that

$$y' \frac{\partial F}{\partial y'} - F = \text{constant}$$

this is equation (1.22).

Now, applying the special form of Euler's equation to the brachistochrone problem we have:

$$y' \frac{y'}{\sqrt{y(1 + y'^2)}} - \sqrt{\frac{1 + y'^2}{y}} = \text{constant} = \text{[redacted]} \frac{1}{c}$$

or:

$$\frac{1}{\sqrt{y(1 + y'^2)}} = -\frac{1}{c}$$

Solving for y we obtain:

$$\frac{dy}{dx} \equiv y' = \sqrt{\frac{C^2 - y}{y}} \qquad (1.23)$$

thus:

$$x = \int \sqrt{\frac{y}{C^2 - y}}\, dy$$

To integrate this equation, make the substitution

$$y \triangleq \tfrac{1}{2} C^2 (1 - \cos t) \qquad (1.24)$$

So
$$\sqrt{\frac{y}{C^2 - y}} = \sqrt{\frac{\frac{1}{2} C^2 (1 - \cos t)}{\frac{1}{2} C^2 (1 + \cos t)}}$$

but
$$\sqrt{\tfrac{1}{2}(1 - \cos t)} \equiv \sin \tfrac{1}{2} t$$

and
$$\sqrt{\tfrac{1}{2}(1 + \cos t)} \equiv \cos \tfrac{1}{2} t$$

hence
$$\sqrt{\frac{y}{C^2 - y}} \equiv \tan \tfrac{1}{2} t$$

thus:

$$x = \int \tan \frac{t}{2} \frac{dy}{dt}\, dt \qquad (1.25)$$

or:
$$x = \int \tan \frac{t}{2} \left(\frac{C^2}{2} \sin t \right) dt$$

Finally:
$$x = C^2 \int \sin^2 \frac{t}{2}\, dt$$

The Solutions are:

a) $$x = \frac{C^2}{2} (t - \sin t) + \text{constant} \qquad (1.26)$$

b) $$y = \frac{C^2}{2} (1 - \cos t) \qquad (1.24)$$

x and y are given as functions of the parameter, t. The brachistochrones are cycloids described by points on the circumference of a circle of radius $C^2/2$ which rolls on the x axis.

2. Generalizations of the Simple Variational Problems

2.0 Integrals with More Than One Argument Function

Our first variational problem concerned the case where:

$$I\{\phi\} = \int_{x_0}^{x_1} F(x, y, y')\, dx \qquad (1.9)$$

and we were to find the single function of x, $y = \phi(x)$ which caused "I" to assume a stationary value.

Now suppose that there are a number of unknown functions, $y_i + \phi(x)$, to be determined. That is: Let $F(x, y_1, y_2, \ldots y_n, y', \ldots y_n')$ be a function of the $2_n + 1$ arguments, x, $\phi_1, \ldots \phi_n, \phi_1', \ldots \phi_n'$, which is continuous and has continuous first and second derivations. Since the $y_i + \phi\, i\,(x)$ are all functions of the single variable, F is a function of the single independent variable, x, and

$$I\left\{\phi_1, \ldots \phi_n\right\} = \int_{x_0}^{x_1} F\, dx \qquad (1.25)$$

is a definite number; and for a specific set of limits, this number depends on the choice of the functions ϕ_i. Our problem will be to find the particular set of n functions, $y_1 = \phi_i(x)$, which causes the integral "I" to assume a stationary value.

To determine that "I" has reached a stationary value, we must compare
OBTAINED BY SETTING $Y_i = u_i(x)$ WITH THE NUMBER
the number, "I", obtained by setting $y_i = \phi_i(x)$, where $u_i(x)$ is the set of functions which satisfies our problem and the $\phi_i(x)$ are all other functions which have the appropriate continuity properties and which satisfy the boundary conditions:

$$\begin{aligned} \phi_i(x_0) &= A_i = u_i(x_0) \\ \phi_i(x_1) &= B_i = u_i(x_1) \end{aligned} \qquad (1.26)$$

Our procedure in this case is very similar to the procedure used in obtaining Euler's Equation for the simplest problem.

We take $y_i = u_i(x)$ to be the particular set of functions which satisfies our problem. We then imbed this set of functions in a one-parameter family of functions depending on a single parameter, ϵ, as follows:

Let $\eta_1(x), \ldots \eta_n(x)$ be n arbitrary functions of x with continuous first and second derivations, which vanish at $x = x_o$ and $x = x_1$; that is:

$$\eta_i(x_o) = \eta_i(x_1) = 0 \tag{1.27}$$

Let: $\quad y = \phi_i = u_i(x) + \epsilon\,\eta_i(x)$

and: $\quad y' = \phi'_i = u'_i(x) + \epsilon\,\eta'_i(x)$

Then the integral "I" becomes simply a function of ϵ :

$$I(\epsilon) = \psi(\epsilon) = \int_{x_o}^{x_1} F(x, u_1 + \epsilon\eta_1, \ldots u_n + \epsilon\eta_n, u'_1, + \epsilon\eta'_1, \ldots u'_n + \epsilon\eta'_n)\, dx \tag{1.28}$$

or:

$$I(\epsilon) = \psi(\epsilon) = \int_{x_o}^{x_1} F(x; \sum_{i=1}^{n} (u_i + \epsilon\eta_i); \sum_{i=1}^{n} (u'_i + \epsilon\eta'_i)\, dx$$

A necessary condition that "I" be stationary when $y_i = u_i(x)$, (that is, when $\epsilon = 0$), is that

$$\left.\frac{d\psi}{d\epsilon}\right|_{\epsilon=0} = 0 \tag{1.29}$$

then $$\frac{d\psi}{d\epsilon} = 0 = \frac{d}{d\epsilon}\int_{x_o}^{x_1} F(x, \sum_{i=1}^{n} u_i + \epsilon\eta_i; \sum_{i=1}^{n} u'_i + \epsilon\eta'_i)\, dx$$

Interchanging the order of integration and differentiation

$$\frac{d\psi}{d\epsilon} = 0 = \int_{x_o}^{x_1} \sum_{i=1}^{n} (\eta_i \frac{\partial F}{\partial y_i} + \eta'_i \frac{\partial F}{\partial y'_i})\, dx$$

or: $$\frac{d\psi}{d\epsilon} = 0 = \sum_{i=1}^{n} \int_{x_o}^{x_1} (\eta_i \frac{\partial F}{\partial y_i} + \eta'_i \frac{\partial F}{\partial y'_i})\, dx$$

If as before we integrate the second term of each of the above integrals by parts and recall that:

$$\eta_i(x_o) = \eta_i(x_1) = 0$$

we then obtain:

$$\frac{d\psi}{d\epsilon} = 0 = \sum_{i=1}^{n} \int_{x_o}^{x_1} \eta_i \left[\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'_i}\right) \right] dx \tag{1.30}$$

Thus we have a set n similar integrals the sum of which must equal zero.

Now the functions η_i (x) are independently arbitrary, hence, the sum of the integrals in equation (1.30) can equal zero for all arbitrary valves of η_i (x) only if each of the integrals equal zero. Furthermore this then requires that:

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \tag{1.31}$$

For $i = 1$ to n

Hence a necessary and sufficient condition that the integral $I(u_1, \ldots u_n)$ may be stationary is that the n functions u_i (x) shall satisfy the system of Euler's equations:

$$\frac{\partial F}{\partial u_i} - \frac{d}{dx}\frac{\partial F}{\partial u_i'} = 0 \qquad (i = 1 \text{ to } n)$$

This is a system of n second order differential equations for the n functions u_i (x). All solutions of the system of equations are called "Extremals" of the variation problem.

For the special case where "F" does not contain the independent variable, x, explicitly, a first integral of the system of Euler equations is given by

$$F - \sum_{i=1}^{n} u_i' \frac{\partial F}{\partial u_i'} = \text{Constant} \tag{1.32}$$

The proof is very similar to the proof for the simplest case. Differentiation of the above expression with respect to x results in an expression which must be equal to zero if the Euler equations are valid.

Proof:

If equation (1.32) is true, then

$$\frac{d}{dx}\left[F - \sum_{i=1}^{n} u_i' \frac{F}{u_i'}\right] = 0$$

thus

$$\frac{d}{dx}(F) - \sum_{i=1}^{n} \frac{d}{dx}\left(u_i' \frac{\partial F}{\partial u_i'}\right) = 0$$

however, $F = F(u_i, u_i')$

therefore expanding the above equation we obtain:

$$\sum_{i=1}^{n} \left\{ \frac{\partial F}{\partial u_i} u_i' + \frac{\partial F}{\partial u_i'} u_i'' \right\} - \sum_{i=1}^{n} \left\{ u_i'' \frac{\partial F}{\partial u_i'} + \frac{d}{dx}\left(\frac{\partial F}{\partial u_i'} \right) \right\} = o$$

or
$$\sum_{i=1}^{n} u_i \left\{ \frac{\partial F}{\partial u_i} + \frac{d}{dx}\left(\frac{\partial F}{\partial u_i'} \right) \right\} = o$$

The quantity in the brackets,

$$\frac{\partial F}{\partial u_i} + \frac{d}{dx}\left(\frac{\partial F}{\partial u_i'} \right) = o$$

is the set of Euler equations and are equal to zero. Thus equation (1.32) is valid.

2.1 Three Dimensional Brachistochrone

Example: The brachistochrone problem in three dimensions. We again take gravity to act in the positive y direction. The problem now is to minimize the integral

$$I = \int_{x_o}^{x_1} \sqrt{\frac{1 + y'^2 + z'^2}{y}}\, dx \tag{1.32}$$

$$I = \int_{x_o}^{x_1} F(y, y', z')\, dx \tag{1.33}$$

The set of Euler's equations for the problem are:

1) $$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z'} = \frac{d}{dx}\frac{z'}{\sqrt{y}}\frac{1}{\sqrt{1 + y'^2 + z'^2}} = o \tag{1.34}$$

$$\frac{z'}{y}\frac{1}{\sqrt{1 + y'^2 + z'^2}} = a$$

2) $$F - y'\frac{\partial F}{\partial y'} - z'\frac{\partial F}{\partial z'} = b \tag{1.35}$$

$$\frac{1}{\sqrt{y}}\frac{1}{\sqrt{1+y'^2+z'^2}} = b$$

Dividing 1) by 2), we see that $z' = \text{constant} = c = \frac{a}{b}$. Therefore, the curve for which the integral is an extreme must lie in the plane

$$z = cx + d \qquad (1.36)$$

Substitution of this expression into either 1) or 2) results in an equation for y which is formally identical to that obtained in the two-dimensional case. The answer is the equation of a cyclid with undetermined constants sufficient to allow the boundary conditions to be satisfied.

2.3 Variational Problem with Constraints

Very often a variational problem is presented where, in addition to the problem of determining a function which causes an integral to assume a stationary value, the resulting function must satisfy subsidary conditions. That is, the field of functions to be investigated is restricted to functions which satisfy a subsidary condition. An example is the so-called "isoperimetric" problem stated as:

Find the curve, y (x), which encloses the greatest area in the xy plane, and which has a given length.

Here the given length of the curve imposes a constraint. The general problem of this type is to find the particular function, $\phi = U(x)$ which causes

$$I\{\phi\} = \int_{x_0}^{x_1} F(x, \phi, \phi')\, dx \qquad (1.9)$$

to be stationary, ϕ being subject to the further subsidary condition

$$H\{\phi\} = \int_{x_0}^{x_1} G(x, \phi, \phi')\, dx = \text{constant} \qquad (1.37)$$

(I)

Before developing the means for solving (1.9) subject to (1.37), we shall briefly review the subject of maxima and minima of functions of several variables.

(I) This is just one form of a constraint. Contraints may be in the form of differential equations, or in the form of inequaltities, etc.; but we will only consider the integral form of a constraint.

2.3 Maxima and Minima with Subsidiary Conditions

We consider the problem of finding stationary values of a function, $f(x, y)$ when the two "independent" variables are not actually independent, but are related by the subsidary condition:

$$\phi (x, y) = 0 \tag{1.38}$$

This is not a fundamentally new problem, since we can in theory, at best, solve (1.38) for one variable in terms of the other, substitute this expression in f (x, y) which then is a function of one variable only and proved along well-known lines.

It is more convenient and (according to reference a) "also more elegant" to preserve the symmetry of the problem and express the conditions for a stationary value in a way which gives no preference to either variable. A very practical reason for preserving the symmetry is that often the subsidary expression is such that it cannot readily be solved for one variable in terms of the other, or, if solvable, the resulting expression is very clumsy to handle.

The problem may be visualized with the aid of figure 2.

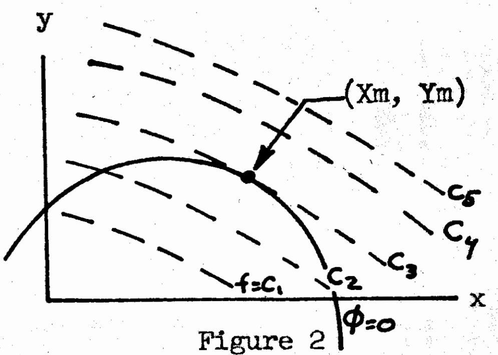

Figure 2

The plane is covered with curves $f(x, y) = c$ which intersect the curve $\phi(x,y) = o$

Each of the curves $f = c_1, c_2\ c_3, c_4$ and c_5 intersect the curve $\phi = o$ and therefore have points on values which satisfy the constraint condition. We desire to know the maximum value of f which satisfies the constraint conditions.

Our problem is therefore, to find the coordinates of the point (Xm, Ym) where f (x,y) has reached an extreme value, (in this case c_3), and simultaneously where $\phi = o$.

At the point (Xm, Ym) the two curves, $f = c$ and $\phi = o$, will have the same tangent. It will be recalled that the slope of a curve

$F(x, y) = c$ is computed as:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = o$$

or:

$$\frac{dy}{dx} = - \frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \qquad (1.39)$$

Thus in the case being considered here, we can write

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = \frac{\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}} \qquad (1.40)$$

at the point Xm, Ym or: If we introduce a constant of proportionately, λ, we have:

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial x}} = \frac{\frac{\partial f}{\partial y}}{\frac{\partial \phi}{\partial y}} \triangleq -\lambda \qquad (1.41)$$

Solving equation (1.41), we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = o \quad \text{and} \quad \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = o \qquad (1.42)$$

Equations (1.42), together with the constraint equation

$$\phi\ (x,y) = o \qquad (1.38)$$

are three equations in the three unknowns, x_m, y_n and λ.

The above discussion is only intended to make the following rule seem plausible. The rule is proven analytically in most calculus books, (for example, reference a).

The factor λ is usually known as Lagranges Multiplier and the following rule is known as Lagranges method of Undetermined Multipliers.

2.5 Lagranges' Method of Undetermined Multipliers.

Lagranges' rule is expressed as, (reference a, page 191), "To find the extreme values of the function f (x,y) subject to the subsidary condition ϕ (x,y) = o, we add to f (x,y) the product of ϕ (x,y) an unknown factor, λ , independent of x and y, and write down the known necessary conditions:

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = o \qquad (1.42)$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = o$$

for an extreme value of

$$F \triangleq f + \lambda \phi \qquad (1.43)$$

Equations (1.42) together with ϕ = o are sufficient to determine the coordinates of the extreme value and the constant of proportionality,

Hence, for the determination of the quantities x_n, x_m and λ this rule gives as many equations as there are unknowns. We have, therefore, replaced the problem of finding the positions of the extremes (x_n, y_n) by a problem in which there is an additional unknown factor, λ , however, now we have the advantage of complete symmetry.

Example:

Find the extreme values of

$$u = xy \tag{1.44}$$

on the unit circle with center at the origin, that is, subject to the constraint

$$\phi (x,y) = x^2 + y^2 - 1 = o \tag{1.45}$$

For this case equation (1.43) becomes

$$F = xy + \lambda (x^2 + y^2 - 1) \tag{1.46}$$

$$\left(\frac{\partial f}{\partial x}\right) = y + 2\lambda x = o$$

$$\left(\frac{\partial f}{\partial y}\right) = x + 2\lambda y = o \tag{1.47}$$

$$\phi = x^2 + y^2 - 1 = o$$

Equations 1.47 may be solved for x, y, λ . We obtain the four points:

$$x = \pm \frac{\sqrt{2}}{2}$$

$$y = \pm \frac{\sqrt{2}}{2}$$

and $\lambda = \pm \frac{1}{2}$ give the

$$u = \pm \frac{1}{2}$$

value of λ and u at the four points. That is, there are four points on the circle where u is stationary - two maxima and two minima.

We will find that Lagrange's method of undetermined multipliers is very useful in handling variational problems with constraints.

2.6 Isoperimetric-type Problems

Returning now to the variational problem with constraints, we have to find the stationary value of

(see following page)

$$I = \int_{x_0}^{x_1} F(x, y, y') \, dx \qquad (1.9)$$

subject to the constraint

$$H = \int_{x_0}^{x_1} G(x, y, y') \, dx = \text{constant} \qquad (1.37)$$

As usual, we assume that $y = u(x)$ is the curve which satisfies our requirements. Again we form a function which is varied from $u(x)$, but in order to allow freedom to satisfy the constraint equation we introduce two parameters, ϵ_1, ϵ_2 and let $u(x)$ be a member of the two parameter family

$$y = u(x) + \epsilon_1 \eta(x) + \epsilon_2 \zeta(x) \qquad (1.38)$$

where η and ζ are twice differentiable and vanish at the end points:

$$\eta(x_1) = \eta(x_0) = \zeta(x_0) = \zeta(x_1) = 0 \qquad (1.39)$$

Now the two integrals are functions of the two parameters, ϵ_1 and ϵ_2

and $$\Psi(\epsilon_1, \epsilon_2) = \int_{x_0}^{x_1} F(x, u + \epsilon_1 \eta + \epsilon_2 \zeta, u' + \epsilon_1 \eta' + \epsilon_2 \zeta') \, dx \qquad (1.40)$$

must be stationary at $\epsilon_1 = \epsilon_2 = 0$ with respect to small values of ϵ_1 and ϵ_2. Where ϵ_1 and ϵ_2 are connected by the relation

$$\phi(\epsilon_1, \epsilon_2) = \int_{x_0}^{x_1} G(x, u + \epsilon_1 \eta + \epsilon_2 \zeta, u' + \epsilon_1 \eta' + \epsilon_2 \zeta') \, dx = \text{constant} \qquad (1.41)$$

But now Lagrange's method of undetermined multipliers tells us, that to find the stationary value of $\Psi(\epsilon_1, \epsilon_2)$ subject to the subsidary condition $\phi(\epsilon_1, \epsilon_2) = \text{constant}$, we should form the function $\Psi + \lambda\phi$ and that

$$\left.\begin{aligned} \frac{\partial \Psi}{\partial \epsilon_1} + \lambda \frac{\partial \phi}{\partial \epsilon_1} &= 0 \\ \frac{\partial \Psi}{\partial \epsilon_2} + \lambda \frac{\partial \phi}{\partial \epsilon_2} &= 0 \end{aligned}\right\} \quad \text{at } \epsilon_1 = \epsilon_2 = 0 \qquad (1.42)$$

are necessary conditions for the existence of a stationary value of Ψ.

The choice y = u (x) assures that the stationary value occurs at

$\epsilon_1 = \epsilon_2 = o$.

Performing the operations indicated by (1.42), and after the usual integration by parts, we obtain

$$\int_{x_o}^{x_1} \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{cx} \frac{\partial F}{\partial y'} \right) + \lambda \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \right\} \eta \, dx = o \qquad (1.43)$$

$$\int_{x_o}^{x_1} \left\{ \left(\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} \right) + \left(\frac{\partial G}{\partial y} - \frac{d}{dx} \frac{\partial G}{\partial y'} \right) \varsigma \, dx = o$$

Now since η and ς are independetly arbitrary and since λ is a constant the integrals of equation (1.43) cannot be zero unless the integrals of each is zero.

Therefore we have that:

$$\frac{\partial F^*}{\partial y} - \frac{d}{dx} \frac{\partial F^*}{\partial y'} = o \qquad (1.44)$$

where $F^* \triangleq F + \lambda G$

The integral of equation 1.44 contains the parameter λ in addition to the two constant os integration. The values of these three constants are determined from the boundary conditions and the constraint equation, (1.37). Examples:

For the isoperimetric problem we seek to maximize the integral:

$$I = \frac{1}{2} \int_{o}^{\pi} r^2 \, d\theta \qquad (1.45)$$

with the constraint

$$l = \int_{A}^{B} ds \qquad (1.46)$$

Here we have the problem of joining two fixed points, A and B by a plane curve of given length, l, so that the area enclosed by the curve and the chord through A and B is maximum.

$$l = \int_A^B ds = \int_0^{\pi} \left\{ r^2 + \left(\frac{dr}{d\theta}\right)^2 \right\}^{\frac{1}{2}} d\theta \qquad (1.47)$$

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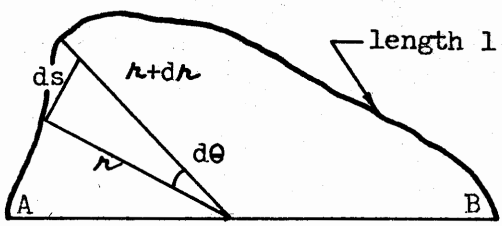

Figure 3

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For this problem:

$$F\ (r,\ r' + \theta\ ) = \tfrac{1}{2} r^2 \qquad (1.48)$$

and

$$G\ (r,\ r',\ \theta) = (r^2 + r'^2)^{\frac{1}{2}} \qquad (1.50)$$

thus for

$$F^* = F + \lambda G \qquad (1.51)$$

we have:

$$F^* = \tfrac{1}{2} r^2 + \lambda (r^2 + r'^2)^{\frac{1}{2}} \qquad (1.52)$$

The Euler equation is then:

$$r + \frac{\lambda r}{\sqrt{r^2 + r'^2)}} - \frac{d}{d\theta}\left\{ \frac{\lambda r'}{\sqrt{r^2 + r'^2}} \right\} = 0 \qquad (1.53)$$

This leads to

$$\frac{-2\left(\frac{\partial r}{\partial \theta}\right)^2 \quad r\frac{\partial^2 r}{\partial \theta^2} - r^2}{\left\{ r^2 + \left(\frac{\partial r^2}{\partial \theta}\right)\right\}^{3/2}} = \frac{1}{\lambda} \qquad (1.54)$$

The left side of equation (1.54) is the expression for the curvature in polar coordinates. Equation (1.54) therefore states that the curvature, $\frac{1}{\rho}$, is to be constant, and the required curve is a circular arc of radius λ passing through the two end points.

Problem:

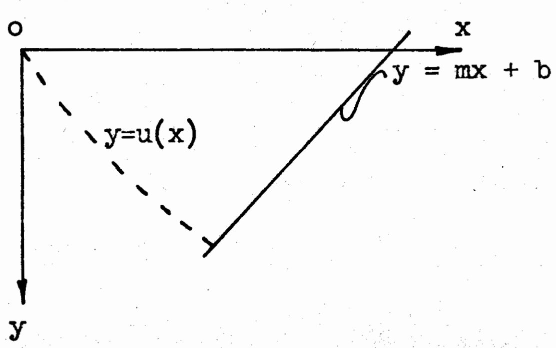
Prove that the sphere is the solid of revolution which, for a given surface area, has maximum volume. Take

$$\text{Area, } A = 2\pi \int y\, ds = 2\pi \int_0^a y\,(1 + y\,x)^2\, dx$$

$$\text{Volume } V = \pi \int_0^a y^2\, dx$$

2.7 The Variational Problem with Variable Limites of Integration

Up to now we have considered only variational problems which required the extremels to pass through fixed end points. Now consider the brachistochrone problem where the curve, y (x) is to join a given fixed point, (the origin), and a given line $y = mx + b$



This problem is representative of a class of problems in which the interval of integration is variable. We will confine ourselves to the case of one independent and one dependent variable only.

Let
$$I = \int_{x_a}^{x_b} F\,(x,\, y,\, y')\, dx \qquad (1.55)$$

Consider x_a as fixed but x_b as variable. Where A and B, the end points of the arc of integration have absciss as x_a and x_b, respectively. In finding the conditions for a stationary value of "I" we will vary not only the arc $y = u(x)$ which joins A and B, but we shall also allow B to move along the curve Γ_2 whose equation is

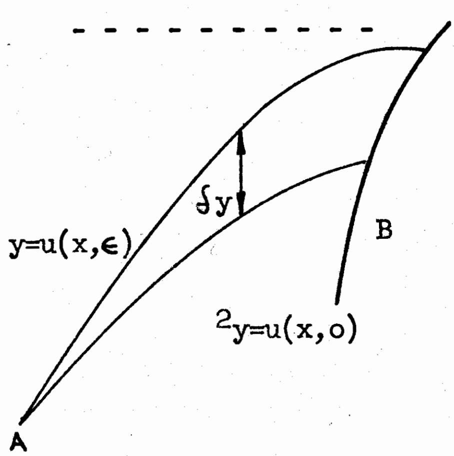
$$y = g_2(x) \tag{1.56}$$

We will keep the point A fixed but the results can be generalized to allow for the case where A can be displaced along

$$y = g_1(x) \tag{1.57}$$

When the end points of the interval were fixed, we varied the function $y(x)$ from the extremal $u(x)$ by constructing the one parameter family $y = u(x) + \epsilon\eta(x)$. Here we will vary $y(x)$ in a more general manner. For the extremal we take $y = u(x, o)$ and for the varied curve we take

$$y = u(x,\epsilon) \tag{1.58}$$



In order to ease our notation problems, we define the symbol δ to be the infinitesimal <u>change</u> of a quantity caused by evaluating that quantity along an admissible curve which neighbors an extremal.

For example:

$$\delta y = u(x,\epsilon) - u(x,o) \tag{1.59}$$

and $$\delta I = \int_a^b F(x, u+\epsilon\eta, u' + \epsilon\eta')dx$$

$$- \int_a^b F\ (x,\ u,\ u')\ dx \qquad (1.60)$$

for <u>fixed</u> end points

Using this notation, the criterion for a stationary value of "I" is:

$$\delta I = o$$

The symbol is called the <u>first</u> <u>variation</u> of the quantity to which it is applied. When the end point B is allowed to vary, we have

$$\delta I = \int_{x_a}^{x_b + dx_b} F\ (x,\ y + \delta y,\ y' + \delta y')\ dx - \int_{x_a}^{x_b} F\ (x,\ y,\ y')\ dx \qquad (1.61)$$

or

$$\delta I = \int_{x_a}^{x_b} \left[F\ (x,\ y + \delta y,\ y' + \delta y') - F(x,\ y,\ y') \right] dx + \int_{x_b}^{x_b + d\,x_b} F\ (x,\ y,\ y')\ dx$$

now: $\delta F = F(x,\ y + \delta y 1 y' + \delta y') - F(x,\ y + \delta y,\ y' + \delta y') = \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y} \delta y'$

and since "δ" in an infinismal change,

$$\int_{x_b}^{x_b + dx_b} F\ (x,\ y,\ y')\ dx = F \Big|_{x=x_b} dx_b$$

Thus

$$\delta I = F \Big|_{x=x_b} dx_b + \int_{x_a}^{x_b} \left| \frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right| dx \qquad (1.62)$$

Now $\delta y' = \delta \frac{dy}{dx} = \frac{d \delta y}{dx}$

Using the identity, the second term in the integral in equation (1.62) can be integrated by parts to give

$$\int_{x_a}^{x_b} \frac{\partial F}{\partial y'} \delta y' \, dx \equiv \left.\frac{\partial F}{\partial y'}\right\}_b \delta y_b - \int_{x_a}^{x_b} \delta y \frac{d\left(\frac{\delta F}{\delta y'}\right)}{dx} \, dx$$

Hence:

$$\delta I = F_b \, dx_b + \left.\frac{\partial F}{\partial y'}\right)_b \delta y_b + \int_{x_a}^{x_b} \delta y \left\{ F_y - \frac{d}{dx} F_{y'} \right\} dx \qquad (1.63)$$

Here; δy_b $u(b,\epsilon) - u(b, o)$ and the subscript b indicates values corresponding to x = b.

Since "I" must be stationary even if x_b is fixed, that is; if the first two terms of (1.63) are zero, we can repeat our earlier argument and conclude that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = o \qquad (1.64)$$

This condition, however, is not sufficient. If B varies, we must in addition require that

$$F_b \, dx_b + \left.\frac{\partial F}{\partial y'}\right\}_b \delta y_b = o \qquad (1.65)$$

Now, from the Figure below

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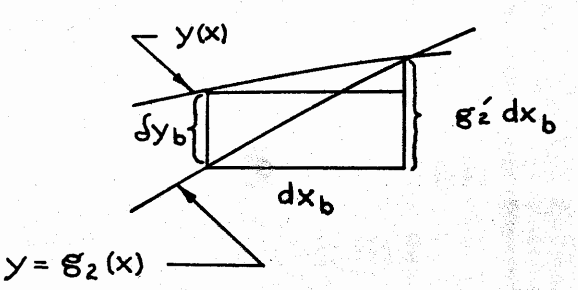

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From the figure, it is apparent that for small dx_b

$$\delta y_b = g_2' \, dx_b - y' \, dx_b \qquad (1.66)$$

So that (1.65) becomes

$$\left[F_b + (g_2' - y') \frac{\partial F}{\partial y'}\Big)_b \right] dx_b = 0 \tag{1.67}$$

Since dx_b is arbitrarily small, the quantity in the brackets must equal zero. Thus, in addition to (1.65) we require

$$\left\{ F + (g_2' - y') \frac{\partial F}{\partial y'}_{\ x=b} \right\} = 0$$

This equation is known as a transversality condition and the curve Γ_2 is said to be transversal to the extremal at B. The same development, of course, can be used when the lower limit of integration is variable. If the end point A can be displaced along the curve Γ_1, where the equation of Γ_1 is $y = g_1(x)$, in addition to B being variable along Γ_2, the following theorem applies: (Reference b, page 213)

Theorem: "If the end points A and B of the range of integration of the integral $I = \int_a^b F(x, y, y') \, dx$ can be displaced along prescribed curves, then I is stationary when the following necessary conditions are satisfied:

1) y, the ordinate of the extremal, satisfies the Eulerian equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx} \frac{\partial F}{\partial y'} = 0 \tag{1}$$

2) at $x=a$, $$F + (g_1' - y') \frac{\partial F}{\partial y'} = 0 \tag{2}$$

where a is the abscissa of the end point A, which can be displaced along the curve $y = g_1(x)$

3) at $x = b$, $$F + (g_2' - y') \frac{\partial F}{\partial y'} = 0 \tag{3}$$

Where b is the abscissa of the end point B, which can be displaced along the curve $y = g_2(x)$.

In these equations, y' is the slope of the extremal and g_1', g_2' are respectively the slopes of the displacement curves of A and B at $x = a$ and $x = b$."

Problem:

$$I = \int_A^B G(x,y)\,(1 + y'^2)^{\frac{1}{2}}\,dx$$

A and B are both variable. Prove that the extremal and Γ_1 intersect orthogonally at A, and that the extremal and Γ_2 intersect orthogonally at B. Orthogonality is shown by $g'\,y' + 1 = 0$

3. Numerical Methods*

3.0 Introduction

There is a relatively small number of analytical solutions to variational problems known. Consequently, numerical methods have been used extensively. When faced with the necessity of solving a variational problem numerically one usually has the choice of attacking the problem directly or of reducing it to a differential equation (Euler's) and solving the differential equation numerically. Since the class of solutions of Euler's equation is enormously restricted compared with the class of all functions which must be tried in the integral equation, and since machine methods of solution of differential equations are well established, it usually is preferrable to deduce the Euler equation and solve it, rather than the original stationary value problem.

There are, however, two methods of attacking the problem directly which are well developed and which are well suited to machine methods:

1. Rayleigh - Ritz Method
2. Galakin Method

These methods will be discussed very briefly. For a more complete discussion, references b and c may be consulted and there is a fairly [redacted] extensive body of literature on direct solution of variational problems.

3.1 The Rayleigh - Ritz Method

The object of this method is to replace the variational problem by that

*Reference (b), Chapter VII

of finding extreme values of functions of several variables.

First it is assumed that y can be expressed in terms of known functions of x. For example, y might be assumed to be expandible in a power series, or in a Fouier series. On substituting the assumed expression for y in the integral, the integral can be evaluated, the coefficients in the expression for y remaining to be evaluated.

By the usual methods of the calculus, the coefficients be adjusted to maximize (or minimize) the integral.

Example:

Minimize the integral

$$I = \int_{-1}^{1} (1 - x^2)\,(y'^2)\,dx \qquad (1.69)$$

Subject to the subsidary condition:

$$\int_{1}^{1} y^2\,dx = 1 \qquad (1.70)$$

Let us assume that y can be expanded in a power series and that the first three terms will give reasonable accuracy for our purposes; thus:

$$y = a + bx + cx^2 \qquad (1.71)$$

Substituting our assumed expression for y in (1.69) and (1.70) we obtain

$$I = \frac{4}{3}\left(b^2 + \frac{4}{5}c^2\right) \qquad (1.72)$$

and

$$1 = 2\left(a^2 + \frac{b^2}{3} + \frac{2ac}{3} + \frac{c^2}{5}\right) \qquad (1.73)$$

(1.72) is a function of three variables, a, b, c (a is missing) which is to be made stationary subject to the constraint equation (1.73). But this is precisely the problem which we treated on pages 14 thru 17 .

Our condition for a stationary value of I is, according to equation (1.41)

$$\frac{\partial I/\partial a}{\partial \phi/\partial a} = \frac{\partial I/\partial b}{\partial \phi/\partial b} = \frac{\partial I/\partial c}{\partial \phi/\partial c} = -\lambda \qquad (1.74)$$

Evaluation of (1.74) results in:

$$\frac{0}{4(a+c/3)} = \frac{8b/3}{4b/3} = \frac{32c/15}{4(a/3+c/5)} = -\lambda \qquad (1.75)$$

The possible solutions are:

1) $\lambda = -2, \; a = c = o \; , \; b = \sqrt{\frac{3}{2}}$

2) $\lambda = o \; , \; b = c = o \; , \; a = \frac{1}{\sqrt{2}}$

3) $\lambda = -6, \; b = o \quad c = -3a, \; a = \sqrt{5/8}$

These are easily seen from the first ratio in equation (1.75).

Finally:

$$-\lambda = o \qquad y = 1/\sqrt{2}$$
$$-\lambda = 2 \qquad y = \sqrt{3/2}\, x \qquad (1.76)$$
$$-\lambda = 6 \qquad y = \sqrt{5/8}\,(1-3x^2)$$

These functions are the first three legendre functions, except for a constant multiplier

$$y = \left(\frac{2n+1}{2}\right)^{\frac{1}{2}} P_n\,(x) \qquad n = 9,\ 1,\ 2 \qquad (1.77)$$

Problem:

Solve the above problem analytically. Hint: Take $\lambda = n\,(n+1)$.

Legendre's equation is:

$$\frac{d}{dx}\left\{(\,1+x^2)\,\frac{dy}{dx}\right\} + n\,(n+1)\,y = o$$

In this particular case, the numerical method results in an exact solution of the variational problem. This, of course, only happens in text-books and for carefully doctored problems. In the real world, the result will be an approximation, the closiness of the result depending on the selection of the approximating functions. For this reason, before attempting a solution by the Rayleigh-Ritz method (or any other numerical method, for that matter) it is highly advisable to carefully investigate the problem analytically so that at least the gross characteristics of the solution are known.

The problem of investigating the degree of approximation achieved is highly important--it also can be very difficult. One indication is the closiness to which the left side of the Eulerian equation approaches zero. If it vanishes identically throughout the interval then the solution is exact.

3.2 Galerkin's Method

The Rayleigh-Ritz method attacks the problem by converting it to an ordinary stationary value problem. The Galerkin method uses the condition for a stationary value, but does not convert the problem.

On page 6, Equation (1.19), we showed that a necessary condition for a stationary value of "I" is that:

$$\int_{x_o}^{x_1} \eta \left(F_u - \frac{d}{dx} F_{u'}\right) dx = o \qquad (1.19)$$

If u is an exact solution of the problem, then (1.19) is true for any arbitrary η which satisfies the conditions of the problem. If u is not an exact solution of Euler's equation, then the quantity in parenthesis (Euler's Expression) does not vanish identically throughout the interval and (1.19) is then not satisfied by arbitrary η.

Let us choose

$$y_n = \sum_{m=1}^{n} a_m f_m (x) \qquad (1.78)$$

as an approximate solution of Euler's equation (1.20). Since (1.19) is true for arbitrary η, we choose:

$$\eta = f_m (x) \qquad m = 1 \ldots n.$$

We then substitute our approximate solution y_n in (1.19) and obtain

$$\int_{x_o}^{x_1} f_m (x) \left\{\frac{\partial F}{\partial y_n} - \frac{d}{dx} \frac{\partial F}{\partial y_n'}\right\} dx = o \qquad (1.79)$$

Because y_n is not an exact solution of Euler's equation, the integrand does not, in general, vanish. In fact it contains n arbitrary constants, a_m, and after the integration is carried out, the result is an equation in the n constants, a.

$$g\ (a_., \ a_z\ \ldots a_n) = o \qquad (1.80)$$

Since there are n different functions, f_m, we can determine n equations in the n unknown constants and satisfy equation (1.79). It is not obvious--to me, at least--that this process requires y_n to converge to u, a solution of Euler's equation. Rather than investigating the problem, however, we will take refuge in a well-known device of people who write technical papers and state that such an investigation exceeds the scope of the present work.

Problem: Use Galerkin's method to verify the above example of the Rayleigh-Ritz method.

4.0 Conclusion

In the time, (and space), which we can devote to the calculus of variations, we have been able to give the briefest consideration to a few of the most important problems. Many questions have been ignored or gracefully side-stepped. For example; most of our development has assumed existence of a solution. We have often required the existence of higher derivatives, (a restrictive condition that may not be required by the physics of the problem). We have given no consideration to determining the type of stationary solution which satisfies the Eulerian equation, that is; whether we have found a maximum, minimum, or an inflection point. All of these considerations are of great importance in particular problems.

As an example of the type of problems that can be encountered, consider the problem of joining two points A and B by an extremal of the integral "J" defined as:

$$J = \int_{x_o}^{x_1} (y')^2 \; (y' + 1)^2 \, dx \tag{1.81}$$

The Euler equation for this case is:

$$y'(y' + 1)^2 + y'^2 \,(y' + 1) = \text{constant} \tag{1.82}$$

or

$$y' \,(y' + 1)\,(2y' + 1) = \text{constant} = k$$

The solution is:

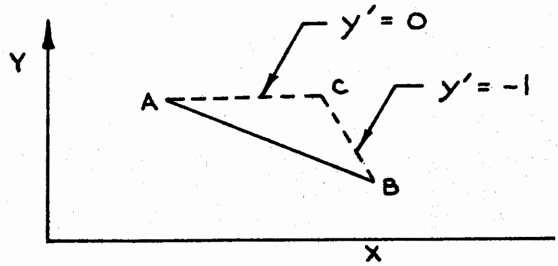
$$y' = \text{constant} = m$$

$$y = mx + b$$

This solution gives a value of "J" as :

$$J = \int_{x_o}^{x_1} m^2 \,(m + 1)^2 \, dx = m^2 \,(m + 1)^2 \left[x_1 - x_o\right]$$

Suppose that A and B are located so that the slope of y is between -1 and o. Then certainly J has a positive value.



But now consider the path ACB where $y' = o$ from A to C, and $y' = -1$ from C to B. Then for this "path" the integrand of J vanishes throughout the interval, hence

$$J \equiv o$$

This discontinuous solution of equation (1.81) gives a smaller value of "J" than the continuous solution "minimum" value of

$$J = m^2 (m + 1)^2 [x_1 - x_0]$$

Thus the solution of the Eulerian equation did not furnish a minimum value in this case as it should have. The solution ACB was ruled out by the requirement for continuity imposed in obtaining Eulers equation.

This rather melancholy situation can be remedied, but the considerations are appreciably more sophisticated than we can handle here. In many cases such solutions can be ruled out on physical grounds--however, there are numerous physical problems where such discontinuous solutions are precisely the ones sought, (for example; the torque curves of an optimum bang-bang servo, and the discontinuous thrust in some trajectory problems).